



On Indexed Riesz Summability of an Infinite Series

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Abstract

Generalizing the results of Seyhan and Misra et al. a theorem on indexed-Reisz summability has been established.

Keywords: $\left|\overline{N}, p_n\right|_k$ Summability; $\varphi - \left|\overline{N}, p_n\right|_k$ Summability; $\varphi - \left|\overline{N}, p_n, \delta\right|_k, k \geq 1$ Summability.

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1. Introduction

Let $\{s_n\}$ denote the n th partial sum of an infinite series $\sum a_n$ and let $\{p_n\}$ be a sequence of positive real constants such that

$$(1.1) \quad P_n = \sum_{\nu=0}^n p_\nu \rightarrow \infty, \text{ for } n=0,1,2,\dots \quad (P_i = p_i = 0, i < 0).$$

Then the sequence-to-sequence transformation given by

$$(1.2) \quad T_n = \frac{1}{P_n} \sum_{\nu=0}^n p_\nu s_\nu,$$

defines the (\overline{N}, p_n) mean of the sequence $\{s_n\}$.

The series $\sum a_n$ is said to be $\left|\overline{N}, p_n\right|_k, k \geq 1$ summable[1] if

$$(1.3) \quad \sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |T_n - T_{n-1}|^k < \infty.,$$

Taking $p_n=1$ for all n , $\left| \overline{N}, p_n \right|_k$ summability reduces to $\left| C, 1 \right|_k$ summability method.

Further, for a sequence $\{ \phi_n \}$ of positive real numbers the series $\sum a_n$ is said to be $\phi - \left| \overline{N}, p_n \right|_k, k \geq 1$, summable if

$$(1.4) \quad \sum_{n=1}^{\infty} \phi_n^{k-1} |T_n - T_{n-1}|^k < \infty..$$

Taking $\phi_n = \frac{P_n}{p_n}$, for all n , $\phi - \left| \overline{N}, p_n \right|_k$ -Summability method reduces to $\left| \overline{N}, p_n \right|_k$ -Summability method. The series $\sum a_n$ is said to be $\phi - \left| \overline{N}, p_n \cdot \delta \right|_k, k \geq 1, \delta \geq 0$, summable if

$$(1.5) \quad \sum_{n=1}^{\infty} \phi_n^{\delta k + k - 1} |T_n - T_{n-1}|^k < \infty.$$

Taking $\delta = 0$, $\phi - \left| \overline{N}, p_n \cdot \delta \right|_k$ -Summability method reduces to $\phi - \left| \overline{N}, p_n \right|_k$ -Summability method.

2. Known theorems:

Concerning with $\left| C, 1 \right|_k$ -Summability of infinite series $\sum a_n$, in 1957 Flett [2] has established the following result. He proved

Theorem-A:

Let σ_n and τ_n denote the $(C, 1)$ mean of the sequence $\{s_n\}$ and $\{na_n\}$ respectively that is

$$(i) \quad \sigma_n = \frac{1}{n+1} \sum_{v=0}^n s_v$$

and

$$(ii) \quad \tau_n = \frac{1}{n+1} \sum_{v=0}^n v a_v.$$

Then the series $\sum a_n$ is summable $\left| C, 1 \right|_k, k \geq 1$, if and only if

$$(2.1) \quad \sum_{n=1}^{\infty} \frac{1}{n} |\tau_n|^k < \infty.$$

Further in 1995, Seyhan [5] extended the result of Flett to $\phi - \left| C, 1 \right|_k$ -summability by establishing

Theorem-B:

Let σ_n and τ_n be as defined in theorem-A and let $\{\varphi_n\}$ be a sequence of positive real numbers. Then the series $\sum a_n$ is summable $\varphi - |C, 1|_k, k \geq 1$, if and only if

$$(2.2) \quad \sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^k} |\tau_n|^k < \infty.$$

In 2010, Mishra et al. [3] have established a theorem similar to theorem-A, by $|\overline{N}, p_n|_k$ -summability method.

They proved:

Theorem-C:

Let $\{t_n\}$ denote the (\overline{N}, p_n) -mean of the sequence $\{na_n\}$ and $\{T_n\}$ be the sequence as defined in (1.2), where $\{p_n\}$ be a sequence of positive real constants satisfying the following conditions:

$$(a) \quad np_n = O(P_n)$$

$$(b) \quad P_n = O(np_n)$$

and

$$(c) \quad n|\Delta p_n| = O(p_n)$$

Then $\sum a_n$ is summable $|\overline{N}, p_n|_k, k \geq 1$, if and only if

$$(2.3) \quad \sum_{n=1}^{\infty} \frac{|t_n|^k}{n} < \infty.$$

Recently Misra et al [4] established a similar theorem for $\varphi - |\overline{N}, p_n|_k, k \geq 1$, summability method. They prove the following:

Theorem-D:

Let $\{T_n\}$ and $\{t_n\}$ denote the sequences of (\overline{N}, p_n) -mean of the sequence $\{s_n\}$ and $\{na_n\}$ respectively. Let $\{\varphi_n\}, \{p_n\}$ be the sequences of positive real constants satisfying the following conditions:

$$(2.4) \quad np_n = O(P_n)$$

$$(2.5) \quad P_n = O(np_n)$$

$$(2.6) \quad n|\Delta p_n| = O(p_n)$$

and

$$(2.7) \quad \frac{\phi_n}{n} = O(1).$$

Then the series $\sum a_n$ is summable $\phi - \left| \overline{N}, p_n \right|_k, k \geq 1$, if and only if

$$(2.8) \quad \sum_{n=1}^{\infty} \frac{\phi_n^{k-1}}{n^k} |t_n|^k < \infty.$$

In what follows, in this paper we prove a similar theorem on $\phi - \left| \overline{N}, p_n, \delta \right|_k, k \geq 1, \delta \geq 0$ summability method. We prove:

3. Main theorem :

Let $\{\phi_n\}, \{p_n\}$ be the sequences of positive real constants such that

$$(3.1) \quad np_n = O(P_n)$$

$$(3.2) \quad P_n = O(np_n)$$

$$(3.3) \quad n|\Delta p_n| = O(p_n)$$

$$(3.4) \quad \{\phi_n^{\delta k + k - 1}\}, k \geq 1, \delta \geq 0, \text{ is monotonically decreasing.}$$

Let $\{T_n\}$ and $\{t_n\}$ denote the sequences of (\overline{N}, p_n) -mean of the sequence $\{s_n\}$ and $\{na_n\}$ respectively. Then the series $\sum a_n$ is summable $\phi - \left| \overline{N}, p_n, \delta \right|_k, k \geq 1, \delta \geq 0$, if and only if

$$(3.5) \quad \sum_{n=1}^{\infty} \frac{\phi_n^{\delta k + k - 1}}{n^k} |t_n|^k < \infty.$$

4. Required Lemma:

We require the following lemma to prove theorem-3.1.

Lemma-4.1[3]:

Let $\{p_n\}$ be a sequence of positive real constants satisfying (i) and (ii) of Theorem-C, then

$$(4.1) \quad p_{n+1} = O(p_n)$$

and

$$(4.2) \quad p_n = O(p_{n+1})$$

holds good.

5. Proof of the Main Theorem:

Sufficient Part (\Leftarrow):

Since $\{t_n\}$ is the (\overline{N}, p_n) -mean of the sequence $\{na_n\}$, we have

$$(5.1) \quad t_n = \frac{1}{P_n} \sum_{v=0}^n p_v v a_v = \frac{1}{P_n} \sum_{v=1}^n p_v v a_v$$

Then

$$(5.2) \quad \begin{aligned} P_n t_n - P_{n-1} t_{n-1} &= n p_n a_n \\ \Rightarrow a_n &= \frac{P_n t_n - P_{n-1} t_{n-1}}{n p_n} \end{aligned}$$

Now we have

$$\begin{aligned} T_n &= \frac{1}{P_n} \sum_{v=0}^n p_v s_v \\ &= \frac{1}{P_n} \sum_{v=0}^n p_v \sum_{\lambda=0}^v a_\lambda \\ &= \frac{1}{P_n} \sum_{\lambda=0}^n a_\lambda \sum_{v=\lambda}^n p_v \\ &= \frac{1}{P_n} \sum_{\lambda=0}^n a_\lambda (P_n - P_{\lambda-1}) \\ &= \sum_{\lambda=0}^n a_\lambda - \frac{1}{P_n} \sum_{\lambda=1}^n a_\lambda P_{\lambda-1} \end{aligned}$$

Then

$$\begin{aligned} (5.3) \quad \nabla T_n &= T_n - T_{n-1} \\ &= \sum_{\lambda=0}^n a_\lambda - \frac{1}{P_n} \sum_{\lambda=1}^n a_\lambda P_{\lambda-1} - \sum_{\lambda=0}^{n-1} a_\lambda + \frac{1}{P_{n-1}} \sum_{\lambda=1}^{n-1} a_\lambda P_{\lambda-1} \\ &= a_n + \left(\frac{1}{P_{n-1}} - \frac{1}{P_n} \right) \sum_{\lambda=1}^{n-1} a_\lambda P_{\lambda-1} - \frac{P_{n-1} a_n}{P_n} \\ &= a_n \left(1 - \frac{P_{n-1}}{P_n} \right) + \frac{P_n}{P_n P_{n-1}} \sum_{\lambda=1}^{n-1} a_\lambda P_{\lambda-1} \\ &= \frac{P_n}{P_n P_{n-1}} \sum_{\lambda=1}^n a_\lambda P_{\lambda-1} \end{aligned}$$

Using (5.2) we get

$$\nabla T_n = \frac{P_n}{P_n P_{n-1}} \sum_{v=1}^n P_{v-1} \frac{P_v t_v - P_{v-1} t_{v-1}}{v p_v}$$

$$\begin{aligned}
 &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^n \frac{P_{v-1} P_v t_v}{v p_v} - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^n \frac{P_{v-1}^2 t_{v-1}}{v p_v} \\
 &= \frac{t_n}{n} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \frac{P_{v-1} P_v t_v}{v p_v} - \frac{p_n}{P_n P_{n-1}} \sum_{v=0}^{n-1} \frac{P_v^2 t_v}{(v+1) p_{v+1}} \\
 &= \frac{t_n}{n} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v t_v \left(\frac{P_{v-1}}{v p_v} - \frac{P_v}{(v+1) p_{v+1}} \right) \\
 &= \frac{t_n}{n} - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \frac{P_v t_v}{v} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v^2 t_v \left(\frac{1}{v p_v} - \frac{1}{(v+1) p_{v+1}} \right) \\
 &= \frac{t_n}{n} - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \frac{P_v t_v}{v} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \frac{P_v^2 t_v}{v(v+1) p_{v+1}} \\
 &\quad + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \frac{P_v^2 t_v}{v} \left(\frac{1}{p_v} - \frac{1}{p_{v+1}} \right) \\
 &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \text{ (Say) .}
 \end{aligned}$$

In order to complete the proof of the sufficient part, by using Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \phi_n^{\delta k+k-1} |T_{n,i}|^k < \infty \text{ for } i=1,2,3,4.$$

Now we have

$$\sum_{n=1}^{\infty} \phi_n^{\delta k+k-1} |T_{n,1}|^k = \sum_{n=1}^{\infty} \frac{\phi_n^{\delta k+k-1}}{n^k} |t_n|^k < \infty. \text{ By (3.5).}$$

Next we have

$$\begin{aligned}
 \sum_{n=2}^{m+1} \phi_n^{\delta k+k-1} |T_{n,2}|^k &\leq \sum_{n=2}^{m+1} \phi_n^{\delta k+k-1} \left(\frac{p_n}{P_n P_{n-1}} \right)^k \left(\sum_{v=1}^{n-1} \frac{P_v |t_v|}{v} \right)^k \\
 &= O(1) \sum_{n=2}^{m+1} \left(\frac{\phi_n p_n}{P_n} \right)^{\delta k+k-1} \left(\frac{p_n}{P_n} \right)^{\delta k} \frac{p_n}{P_n P_{n-1}^k} \left(\sum_{v=1}^{n-1} |t_v| p_v \right)^k, \text{ using (3.2)} \\
 &= O(1) \sum_{n=2}^{m+1} \left(\frac{\phi_n p_n}{P_n} \right)^{\delta k+k-1} \left(\frac{p_n}{P_n} \right)^{\delta k} \frac{p_n}{P_n P_{n-1}^k} \left(\sum_{v=1}^{n-1} p_v \right)^{k-1} \left(\sum_{v=1}^{n-1} p_v |t_v|^k \right)
 \end{aligned}$$

Using Holder's inequality

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{\phi_n p_n}{P_n} \right)^{\delta k+k-1} \left(\frac{p_n}{P_n} \right)^{\delta k} \frac{p_n}{P_n P_{n-1}^k} \left(\sum_{v=1}^{n-1} |t_v|^k p_v \right)$$

$$\begin{aligned}
 &= O(1) \sum_{v=1}^m p_v |t_v|^k \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} \left(\frac{\varphi_n p_n}{P_n} \right)^{\delta k + k - 1} \left(\frac{P_n}{P_n} \right)^{\delta k} \\
 &= O(1) \sum_{v=1}^m p_v |t_v|^k \left(\frac{\varphi_v p_v}{P_v} \right)^{\delta k + k - 1} \left(\frac{P_v}{P_v} \right)^{\delta k} \sum_{n=v+1}^{m+1} \left(\frac{1}{P_{n-1}} - \frac{1}{P_n} \right), \text{ using (3.4)} \\
 &= O(1) \sum_{v=1}^m p_v |t_v|^k \left(\frac{\varphi_v p_v}{P_v} \right)^{\delta k + k - 1} \left(\frac{P_v}{P_v} \right)^{\delta k} \frac{1}{P_v} \\
 &= O(1) \sum_{v=1}^m \left(\frac{\varphi_v p_v}{P_v} \right)^{\delta k + k - 1} \left(\frac{P_v}{P_v} \right)^{\delta k - 1} |t_v|^k \\
 &= O(1) \sum_{v=1}^m \frac{\varphi_v^{\delta k + k - 1}}{P_v^k} |t_v|^k, \text{ using (3.2)} \\
 &= O(1), \text{ using (3.5)}
 \end{aligned}$$

Further we have

$$\begin{aligned}
 \sum_{n=2}^{m+1} \varphi_n^{\delta k + k - 1} |T_{n,3}|^k &\leq \sum_{n=2}^{m+1} \varphi_n^{\delta k + k - 1} \left(\frac{p_n}{P_n P_{n-1}} \right)^k \left(\sum_{v=1}^{n-1} \frac{P_v^2 |t_v|}{v(v+1) p_{v+1}} \right)^k \\
 &\leq \sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{\delta k + k - 1} \left(\frac{P_n}{P_n} \right)^{\delta k} \frac{p_n}{P_n P_{n-1}^k} \left(\sum_{v=1}^{n-1} \frac{P_v p_{v+1} |t_v|}{v(v+1) p_{v+1}} \right)^k \\
 &\leq \sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{\delta k + k - 1} \left(\frac{P_n}{P_n} \right)^{\delta k} \frac{p_n}{P_n P_{n-1}^k} \left(\sum_{v=1}^{n-1} |t_v| p_v \right)^k, \text{ using (3.2)} \\
 &= O(1) \text{ as } m \rightarrow \infty, \text{ proceeding as above.}
 \end{aligned}$$

Next we have

$$\begin{aligned}
 \sum_{n=2}^{m+1} \varphi_n^{\delta k + k - 1} |T_{n,4}|^k &\leq \sum_{n=2}^{m+1} \varphi_n^{\delta k + k - 1} \left(\frac{p_n}{P_n P_{n-1}} \right)^k \left(\sum_{v=1}^{n-1} \frac{P_v^2 |t_v|}{v} \left| \frac{1}{p_v} - \frac{1}{p_{v+1}} \right| \right)^k \\
 &\leq \sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{\delta k + k - 1} \left(\frac{P_n}{P_n} \right)^{\delta k} \frac{p_n}{P_n P_{n-1}^k} \left(\sum_{v=1}^{n-1} \frac{P_v^2 |t_v|}{v} \left| \frac{p_{v+1} - p_v}{p_{v+1} p_v} \right| \right)^k, \\
 &= O(1) \sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{\delta k + k - 1} \left(\frac{P_n}{P_n} \right)^{\delta k} \frac{p_n}{P_n P_{n-1}^k} \left(\sum_{v=1}^{n-1} v |\Delta p_v| |t_v| \right)^k, \text{ Using (3.2)}
 \end{aligned}$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{\varphi_n P_n}{P_n} \right)^{\delta k + k - 1} \left(\frac{P_n}{P_n} \right)^{\delta k} \frac{P_n}{P_n P_{n-1}^k} \left(\sum_{v=1}^{n-1} p_v |t_v| \right)^k, \text{ Using (3.3)}$$

$$= O(1) \text{ as } m \rightarrow \infty, \text{ proceeding as above}$$

This proves the sufficient part of the theorem.

Necessary Part (\Rightarrow):

From (5.3) we have

$$\frac{P_{n-1} P_n}{P_n} \nabla T_n = \sum_{v=1}^n P_{v-1} a_v$$

$$a_n = \frac{1}{P_{n-1}} \left[\frac{P_{n-1} P_n}{P_n} \nabla T_n - \frac{P_{n-2} P_{n-1}}{P_{n-1}} \nabla T_{n-1} \right]$$

$$= \frac{P_n}{P_n} \nabla T_n - \frac{P_{n-2}}{P_{n-1}} \nabla T_{n-1}$$

Now

$$t_n = \frac{1}{P_n} \sum_{v=1}^n p_v \nabla a_v$$

$$= \frac{1}{P_n} \sum_{v=1}^n \left(\nu P_v \nabla T_v - \frac{\nu P_v P_{v-2}}{P_{v-1}} \nabla T_{v-1} \right)$$

$$= \frac{1}{P_n} \sum_{v=1}^n \nu P_v \nabla T_v - \frac{1}{P_n} \sum_{v=0}^{n-1} \frac{(\nu+1) P_{v+1} P_{v-1}}{P_v} \nabla T_v$$

$$= n \nabla T_n + \frac{1}{P_n} \sum_{v=1}^n \nu P_v \nabla T_v - \frac{1}{P_n} \sum_{v=1}^{n-1} \frac{(\nu+1) P_{v+1} P_{v-1}}{P_v} \nabla T_v$$

$$= n \nabla T_n + \frac{1}{P_n} \sum_{v=1}^n \nu \nabla T_v (P_v - P_{v-1})$$

$$+ \frac{1}{P_n} \sum_{v=1}^{n-1} \nu \nabla T_v P_{v-1} \left(1 - \frac{P_{v+1}}{P_v} \right) + \frac{1}{P_n} \sum_{v=1}^{n-1} P_{v-1} \frac{P_{v+1}}{P_v} \nabla T_v$$

$$= t_{n,1} + t_{n,2} + t_{n,3} + t_{n,4}, \text{ say.}$$

To complete the necessary part, using Minokowski's inequality, we need to show only

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{\delta k+k-1}}{n^k} |t_{n,i}|^k < \infty. \text{ for } i = 1, 2, 3, 4.$$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\varphi_n^{\delta k+k-1}}{n^k} |t_{n,1}|^k &= \sum_{n=1}^{\infty} \frac{\varphi_n^{\delta k+k-1} n^k}{n^k} |\nabla T_n|^k \\ &= \sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} |\nabla T_n|^k \\ &= O(1). \end{aligned}$$

Further,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\varphi_n^{\delta k+k-1}}{n^k} |t_{n,2}|^k &\leq \sum_{n=2}^{m+1} \frac{\varphi_n^{\delta k+k-1}}{n^k} \frac{1}{P_n^k} \left(\sum_{\nu=1}^{n-1} \nu |\nabla T_{\nu}| p_{\nu} \right)^k \\ &\leq \sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{\delta k+k-1} \left(\frac{P_n}{P_n} \right)^{\delta k} \frac{P_n}{P_n^k P_{n-1}} \left(\sum_{\nu=1}^{n-1} \nu |\nabla T_{\nu}| p_{\nu} \right)^k \\ &= O(1) \sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{\delta k+k-1} \left(\frac{P_n}{P_n} \right)^{\delta k} \frac{P_n}{P_n^k P_{n-1}} \left(\sum_{\nu=1}^{n-1} p_{\nu} \right)^{k-1} \left(\sum_{\nu=1}^{n-1} \nu^k |\nabla T_{\nu}|^k p_{\nu} \right), \end{aligned}$$

Using Holder's inequality

$$\begin{aligned} &= O(1) \sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{\delta k+k-1} \left(\frac{P_n}{P_n} \right)^{\delta k} \frac{P_n}{P_n P_{n-1}} \left(\sum_{\nu=1}^{n-1} \nu^k |\nabla T_{\nu}|^k p_{\nu} \right) \\ &= O(1) \sum_{\nu=1}^m p_{\nu} \nu^k |\nabla T_{\nu}|^k \sum_{n=\nu+1}^{m+1} \left(\frac{1}{P_{n-1}} - \frac{1}{P_n} \right) \left(\frac{\varphi_n p_n}{P_n} \right)^{\delta k+k-1} \left(\frac{P_n}{P_n} \right)^{\delta k} \\ &= O(1) \sum_{\nu=1}^m p_{\nu} \nu^k |\nabla T_{\nu}|^k \sum_{n=\nu+1}^{m+1} \left(\frac{1}{P_{n-1}} - \frac{1}{P_n} \right) (\varphi_n)^{\delta k+k-1} \left(\frac{P_n}{P_n} \right)^{k-1} \\ &= O(1) \sum_{\nu=1}^m p_{\nu} \nu^k |\nabla T_{\nu}|^k \left(\frac{1}{P_{\nu}} - \frac{1}{P_{m+1}} \right) (\varphi_{\nu})^{\delta k+k-1} \left(\frac{p_{\nu}}{P_{\nu}} \right)^{k-1} \\ &\leq O(1) \sum_{\nu=1}^m \frac{p_{\nu}}{P_{\nu}} \nu^k |\nabla T_{\nu}|^k (\varphi_{\nu})^{\delta k+k-1} \left(\frac{p_{\nu}}{P_{\nu}} \right)^{k-1} \text{ using (3.4)} \end{aligned}$$

$$= O(1) \sum_{\nu=1}^m \varphi_{\nu}^{\delta k+k-1} |\nabla T_{\nu}|^k$$

$$= O(1) \text{ as } m \rightarrow \infty, \text{ using (3.5)}$$

Again

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{\delta k+k-1}}{n^k} |t_{n,3}|^k \leq \sum_{n=2}^{m+1} \frac{\varphi_n^{\delta k+k-1}}{n^k} \frac{1}{P_n^k} \left(\sum_{\nu=1}^{n-1} \nu P_{\nu-1} |\nabla T_{\nu}| \frac{|p_{\nu} - p_{\nu-1}|}{p_{\nu}} \right)^k$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{\varphi_n P_n}{P_n} \right)^{\delta k+k-1} \left(\frac{P_n}{p_n} \right)^{\delta k} \frac{1}{n P_n^k} \left(\sum_{\nu=1}^{n-1} \nu P_{\nu-1} |\nabla T_{\nu}| \frac{|\Delta p_{\nu}|}{p_{\nu}} \right)^k,$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{\varphi_n P_n}{P_n} \right)^{\delta k+k-1} \left(\frac{P_n}{p_n} \right)^{\delta k} \frac{1}{n P_n^k} \left(\sum_{\nu=1}^n P_{\nu-1} |\nabla T_{\nu}| \right)^k, \text{ using (3.3)}$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{\varphi_n P_n}{P_n} \right)^{\delta k+k-1} \left(\frac{P_n}{p_n} \right)^{\delta k} \frac{1}{n P_n^k} \left(\sum_{\nu=1}^n P_{\nu} |\nabla T_{\nu}| \right)^k$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{\phi_n P_n}{P_n} \right)^{\delta k+k-1} \left(\frac{P_n}{p_n} \right)^{\delta k} \frac{n^{k-1}}{P_n^k} \left(\sum_{\nu=1}^n p_{\nu} |\nabla T_{\nu}| \right)^k, \text{ using (3.2)}$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{\phi_n P_n}{P_n} \right)^{\delta k+k-1} \left(\frac{P_n}{p_n} \right)^{\delta k} \frac{n^{k-1}}{P_n^k} \left(\sum_{\nu=1}^n p_{\nu} \right)^{k-1} \left(\sum_{\nu=1}^n p_{\nu} |\nabla T_{\nu}|^k \right)$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{\phi_n P_n}{P_n} \right)^{\delta k+k-1} \left(\frac{P_n}{p_n} \right)^{\delta k} \frac{n^{k-1}}{P_n} \left(\sum_{\nu=1}^n p_{\nu} |\nabla T_{\nu}|^k \right)$$

$$= O(1) \sum_{\nu=1}^m p_{\nu} |\nabla T_{\nu}|^k \sum_{n=\nu+1}^{m+1} (\phi_n)^{\delta k+k-1} \left(\frac{p_n}{P_n} \right)^{k-1} \frac{n^{k-1}}{P_n}$$

$$= O(1) \sum_{\nu=1}^m p_{\nu} |\nabla T_{\nu}|^k (\phi_{\nu})^{\delta k+k-1} \frac{\nu}{P_{\nu}} \text{ using (3.4)}$$

$$= O(1) \sum_{\nu=1}^m \varphi_{\nu}^{\delta k+k-1} |\nabla T_{\nu}|^k$$

$$= O(1) \text{ as } m \rightarrow \infty, \text{ using (3.5).}$$

Finally,

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \frac{\varphi_n^{\delta k+k-1}}{n^k} |t_{n,4}|^k \\
 & \leq \sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{\delta k+k-1} \left(\frac{P_n}{p_n} \right)^{\delta k} \frac{1}{n P_n^k} \left(\sum_{\nu=1}^{n-1} P_{\nu-1} \frac{P_{\nu+1}}{P_{\nu}} |\nabla T_{\nu}| \right)^k \\
 & = O(1) \sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{\delta k+k-1} \left(\frac{P_n}{p_n} \right)^{\delta k} \frac{1}{n P_n^k} \left(\sum_{\nu=1}^{n-1} P_{\nu-1} |\nabla T_{\nu}| \right)^k, \text{ by lemma} \\
 & = O(1) \sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{\delta k+k-1} \left(\frac{P_n}{p_n} \right)^{\delta k} \frac{1}{n P_n^k} \left(\sum_{\nu=1}^{n-1} P_{\nu} |\nabla T_{\nu}| \right)^k \\
 & = O(1) \text{ as } m \rightarrow \infty, \text{ proceeding as above.}
 \end{aligned}$$

This completes the proof of the theorem.

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